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WAVE ADIABATIC CURVES FOR MEDIA WITH ARBITRARY STATE EQUATION*

A.D. SIDORENKO

Proofs are given of the assertions in /1/, on the basis of which the uniqueness is proved of the solutions of the Riemann problem on the disintegration of an arbitrary discontinuity.

1. Fundamental relations. We consider the system of Hugoniot conditions

$$\varepsilon - \varepsilon_0 = {}^1/_2\,(p + p_0)(v_0 - v),\quad (u - u_0)^2 = (p - p_0)(v_0 - v),\quad D^2 = (p - p_0)/(v_0 - v) \tag{1.1}$$

We assume the fulfillment of the general requirements of thermodynamics /2/

$$d\varepsilon = -p\,dv + T\,ds$$
$$\frac{\partial^2\varepsilon}{\partial v^2} > 0,\quad \frac{\partial^2\varepsilon}{\partial s^2} > 0,\quad \frac{\partial^2\varepsilon}{\partial s^2}\frac{\partial^2\varepsilon}{\partial v^2} - \left(\frac{\partial^2\varepsilon}{\partial s\,\partial v}\right)^2 \geqslant 0 \tag{1.2}$$

The inequalities in (1.2) can be written as

$$\partial p / \partial v < 0,\ T / c_v > 0,\ -(T / c_v)\,\partial p / \partial v \geqslant (\partial p / \partial s)^2 \tag{1.3}$$

A wave adiabatic curve is a continuous direct curve in the space of variables (v, s, p), consisting of admissible segments of the Hugoniot adiabatic curve centered at the point (v_0, s_0, p_0), of the Poisson adiabatic curves and of the enveloping Hugoniot adiabatic curves. The admissible segments are chosen by starting off from the requirement that the discontinuities be stable. On the set of points of the Hugoniot adiabatic curves occurring in the wave adiabatic we require the fulfillment of the inequalities

$$(p - p_0)(v_0 - v) < 4c_vT,\ \ v \leqslant v_0 \tag{1.4}$$
$$\partial p / \partial s > -2T / (v - v_0),\ v > v_0 \tag{1.5}$$

2. General properties of curves H and R. By π_s and π_p we denote coordinate planes of the variables v, s and v, p. In these planes we denote a Hugoniot adiabatic curve H by H_s and H_p, a Poisson adiabatic curve P by P_s and P_p, the curves R by R_s and R_p, and the wave adiabatic curve W by W_s and W_p.

Theorem 1. The points of $H_s \subset W_s$ are not singular.

Proof. Consider the function

$$f(v, s) = \varepsilon - \varepsilon_0 - {}^1/_2\,(p + p_0)(v_0 - v)$$

If a point is singular, then $\partial f / \partial v = 0$ and $\partial f / \partial s = 0$, i.e.,

$$-(\partial p / \partial v)(v_0 - v) - (p - p_0) = 0,\ T - {}^1/_2\,(\partial p / \partial s)(v_0 - v) = 0 \tag{2.1}$$

Having substituted these equalities into the last condition in (1.3), we obtain an inequality contradicting (1.4) when $v \leqslant v_0$. When $v > v_0$ the second equality in (2.1) contradicts (1.5).

We write out the following relations:

$$(2T / (v_0 - v) - \partial p / \partial s)\,ds + (-\partial p / \partial v - (p - p_0) / (v_0 - v))\,dv = 0 \tag{2.2}$$
$$\partial p / \partial v = (p - p_0) / (v - v_0)\ \ (a),\ \ ds / dv = 0\ (b),\ dp / dv = (p - p_0) / (v - v_0)\ (c)$$
$$\partial p / \partial s = 2T / (v_0 - v)\ (d),\ \ dv / ds = 0\ (e),\ \ dp / ds = 2T / (v_0 - v)\ (f),$$
$$-\partial p / \partial v - (p - p_0) / (v_0 - v) > 0,\ \ \partial p / \partial s - 2T / (v_0 - v) < 0,\ \ dv / ds < 0\ (\alpha),$$
$$-\partial p / \partial v - (p - p_0) / (v_0 - v) > 0,\ \ \partial p / \partial s - 2T / (v_0 - v) > 0,\ dv / ds > 0\ (\beta),$$
$$-\partial p / \partial v - (p - p_0) / (v_0 - v) < 0,\ \ \partial p / \partial s - 2T / (v_0 - v) < 0,\ \ dv / ds > 0\ (\gamma),$$
$$-\partial p/\partial v - (p - p_0)/(v_0 - v) < 0,\quad \frac{\partial p}{\partial s} - 2T/(v_0 - v) > 0,\quad dv/ds < 0\ (\delta)$$

First of all, by a differentiation and a substitution of the thermodynamics relations, we can

*Prikl.Matem.Mekhan.,46,No.2,pp.241-247,1982 achieve that equality (2.2) is a consequence of (1.1). Further, the following logical relations are obvious:

$$(2.2) \wedge \neg (d) \Rightarrow ((a) \Leftrightarrow (b) \Leftrightarrow (c)), \quad (2.2) \wedge \neg (a) \Rightarrow ((d) \Leftrightarrow (e) \Leftrightarrow (f)), \quad (2.2) \Rightarrow (\alpha) \vee (\beta) \vee (\gamma) \vee (\delta).$$

We partition π_s into the regions $\Omega_\alpha, \Omega_\beta, \Omega_\gamma, \Omega_\delta$ such that in each of these regions there are fulfilled, respectively, the first two inequalities of systems $(\alpha), (\beta), (\gamma), (\delta)$. We denote the boundaries of these regions by $\Gamma_\alpha, \Gamma_\beta, \Gamma_\gamma, \Gamma_\delta$, respectively.

Lemma 1. When $v \leqslant v_0$

$$(v,s) \notin \Omega_\delta, (v, s) \notin \Gamma_\beta \cap \Gamma_\gamma$$

holds for the points $(v, s) \in H_s$.

Proof. If $(v, s) \in \Omega_\delta$ or $(v, s) \in \Gamma_\beta \cap \Gamma_\gamma$, then at this point we have

$$-\partial p / \partial v - (p - p_0) / (v_0 - v) \leqslant 0 \quad \text{and} \quad \partial p / \partial s - 2T / (v_0 - v) \geqslant 0$$

We obtain a contradiction with (1.4) after these relations are substituted into the last condition in (1.3).

Theorem 2. Inequality (1.4) is sufficient for $d(u - u_0)^2 / ds > 0$ in regions $\Omega_\alpha, \Omega_\beta$.

Proof. From the second and third relations in (1.1) we have $(u - u_0)^2 = D^2 (v_0 - v)^2$. Differentiating this equality, we have

$$d(u - u_0)^2 / ds = (dD^2 / ds)(v_0 - v)^2 - 2(v_0 - v) D^2 dv/ds$$

From the third relation in (1.1) follows

$$dD^2 / ds = (p - p_0) / (v_0 - v)^2 \, dv / ds + (\partial p / \partial v)(v_0 - v) \, dv / ds + (\partial p / \partial s)(v_0 - v)$$

Substituting (2.2) into this equality, we obtain $dD^2 / ds = 2T / (v_0 - v)^2$. From (1.1), (2,2) and the equalities obtained we have

$$d(u - u_0)^2 / ds = (-\partial p / \partial v - (p - p_0) T^{-1} \partial p / \partial s + (p - p_0) / (v_0 - v)) \, 2T / (-\partial p / \partial v - (p - p_0) / (v_0 - v))$$

Allowing for (1.3), we can write

$$-\partial p / \partial v - (p - p_0) T^{-1} \partial p / \partial s + (p - p_0) / (v_0 - v) \geqslant c_v T^{-1} ((\partial p / \partial s)^2 - (p - p_0) c_v^{-1} \partial p / \partial s + T c_v^{-1} \times (p - p_0 / (v_0 - v))$$

For the discriminant Δ of the quadratic trinomial we have

$$\Delta = D^2 c_v^{-2} [(p - p_0) / (v_0 - v) - 4 c_v T]$$

In view of (1.4), $\Delta < 0$; therefore, the right-hand side in the last equality is positive in $\Omega_\alpha, \Omega_\beta$, which proves the theorem.

Let the quantity $v_0 = \bar{v}$ be a variable. Then relations (1.1) define a one-parameter family of Hugoniot adiabatic curves. Functions of $\bar{v}$ and s_0 are designated by an overbar. We require that the relations

$$(d\bar{u})^2 = -d\bar{p} d\bar{v}, \; D^2 = -d\bar{p} / d\bar{v} \tag{2.3}$$

be fulfilled on P (when $s = s_0$). The system of equalities (1.1) (with $v_0 = \bar{v}$) and (2.3) defines the curves R.

Theorem 3. The points of $R_s \subset W_s$ are nonsingular when $v \neq \bar{v}$.

Proof. Consider the functions

$$\bar{f}(v, s, \bar{v}) = \varepsilon - \bar{\varepsilon} - {}^1/_2 (p + \bar{p})(\bar{v} - v), \quad \bar{\varphi}(v, s, \bar{v}) = p - \bar{p} - \bar{p}'(v - \bar{v})$$

For them we write out the Jacobians

$$\partial(\bar{f}, \bar{\varphi}) / \partial(v, \bar{v}) = \bar{p}''(\bar{v} - v)(-{}^1/_2 (\partial p / \partial v)(\bar{v} - v) - {}^1/_2 (p - \bar{p}))$$
$$\partial(\bar{f}, \bar{\varphi}) / \partial(s, \bar{v}) = \bar{p}''(\bar{v} - v)(T - {}^1/_2 (\partial p / \partial s)(\bar{v} - v)) \tag{2.4}$$

In view of (1.4) and (1.5) the Jacobians do not vanish simultaneously when $v \neq \bar{v}$, which proves the theorem.

Lemma 2. The curves R_s are enveloping families of shock adiabatic curves.

Proof. Having differentiated the first equality in (1.1) with $v_0 = \bar{v}$, we obtain

$$d\varepsilon + {}^1/_2\,(p + \bar{p})\,dv - {}^1/_2\,(\bar{v} - v)\,dp = d\bar{\varepsilon} + {}^1/_2\,(p + \bar{p})\,d\bar{v} + {}^1/_2\,(\bar{v} - v)\,d\bar{p}$$

We can convince ourselves that this equality's right-hand side equals zero; therefore, $2Tds - (p - \bar{p})\,dv - (\bar{v} - v)\,dp = 0$. Hence from R_s we have

$$(2T / (\bar{v} - v) - \partial p / \partial s)\,ds + (-\partial p / \partial v - (p - \bar{p}) / (\bar{v} - v))\,dv = 0, \quad v \neq \bar{v} \tag{2.5}$$

Comparing (2.2) and (2.5), we see that the lemma is valid; this yields the possibility of carrying all the fundamental differential properties of the adiabatic curve H over to the curves R, as long as the differential relations with first differentials are analogous.

3. General properties of wave adiabatic curves. We take p as a parameter on the wave adiabatic curve W. At first we consider W when $v \leqslant v_0$. Let $\partial^2 p / \partial v^2 > 0$ at point (v_0, s_0). In a neighborhood of (v_0, s_0) the adiabatic curve $H_s \subset \Omega_\alpha$. Two cases are possible as the pressure increases further:

1) $H_s \subset \Omega_\gamma$, 2) $H_s \subset \Omega_\beta$.

The segment of H_s, satisfying the first condition, should be regarded as inadmissible (nonphysical) since the requirement of stability of discontinuities is violated on it, as we see from inequalities (γ). It is evident that in a neighborhood of the boundary $\Gamma_\alpha \cap \Gamma_\gamma$, by continuity, $s > s_0$, i.e., the requirement of stability of discontinuities is a more stringent constraint than $s > s_0$. Equalities (*a*), (*b*), (*c*) are fulfilled on the boundary $\Gamma_\alpha \cap \Gamma_\gamma$, whence it follows that H_p and P_p have a common tangent, viz., a ray drawn from the initial point (v_0, p_0). From (2.2), on $\Gamma_\alpha \cap \Gamma_\gamma$ we obtain

$$d^n s / dv^n = (\partial^n p / \partial v^n) / (2T / (v_0 - v) - \partial p / \partial s)$$
$$d^n p / dv^n = (2T / (v_0 - v))\,(\partial^n p / \partial v^n)\,(2T / (v_0 - v) - \partial p / \partial s)$$

$n \geqslant 2$ (under the condition $\partial^{n-1} p / \partial v^{n-1} = 0$, $n \neq 2$). Hence we see that H_p and P_p have a like convexity on $\Gamma_\alpha \cap \Gamma_\gamma$. This makes it possible to continue H from the boundary $\Gamma_\alpha \cap \Gamma_\gamma$ in the direction of the growth of p of the adiabatic curve P. The continuation is possible up to a point of inflection at which $\partial^k p / \partial v^k = 0$ $(k = 2, \ldots, 2n)$, $\partial^{2n+1} p / \partial v^{2n+1} \neq 0$. When $v = \bar{v}$, from (2.4) we have

$$\partial^k \bar{\varphi} / \partial \bar{v}^k = 0 \quad (k = 1, \ldots, 2n)$$
$$\partial^{2n+1} \bar{\varphi} / \partial \bar{v}^{2n+1} = 2n\,d^{2n+1} \bar{p} / d\bar{v}^{2n+1} \neq 0$$

and hence $\bar{v} = \bar{v}\,(v, s)$ exists in a neighborhood of this point. Having substituted into (2.5), we obtain the differential equation for R_s. Since $2T - (\partial p / \partial s)\,(\bar{v} - v) \neq 0$ in the neighborhood of point $v = \bar{v}$, it is nonsingular for R_s. From this point there can issue a curve R which too can have nonphysical segments replaced by Poisson adiabatic curves and by new curves R'. The curve R_p intersects H_p at the point (v_3, p_3) (Fig.1).

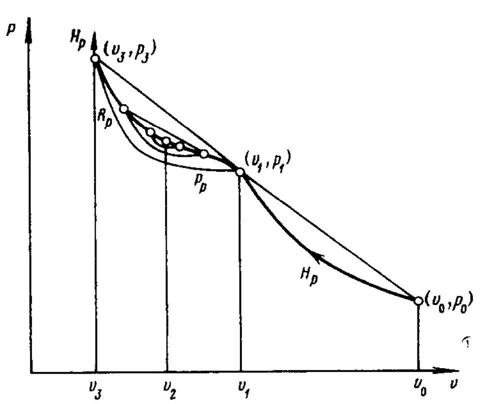

Fig.1

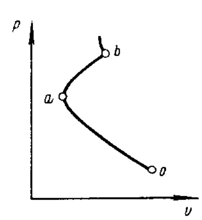

Fig.2

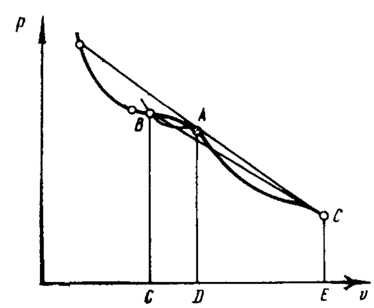

Fig.3

Indeed, $\varepsilon_3 - \varepsilon_1 = {}^1/_2\,(p_3 + p_1)\,(v_1 - v_3)$, $\varepsilon_1 - \varepsilon_0 = {}^1/_2\,(p_1 + p_0)\,(v_0 - v_1)$. The points (v_1, p_1) and (v_3, p_3) lie on a ray drawn from (v_0, p_0) and, therefore, $(p_3 - p_1) / (v_3 - v_1) = (p_1 - p_0) / (v_1 - v_0)$. After this equality has been substituted into the sum of the two preceding, we obtain $\varepsilon_3 - \varepsilon_0 = {}^1/_2\,(p_3 + p_0)\,(v_0 - v_3)$, i.e.,

$(v_3, p_3) \in H_p$.

In the second case, on the strength of Lemma 1, H_s cannot go from Ω_β into Ω_γ through the boundary $\Gamma_\beta \cap \Gamma_\gamma$; therefore, once again H_s goes from Ω_β into Ω_α. In region Ω_α we have $dv/ds < 0$, the equality $dv / ds = 0$ is fulfilled on the boundary $\Gamma_\alpha \cap \Gamma_\beta$, while in Ω_β we have dv/ds $0<$; therefore, H_s and H_p turn when passing through the boundary (Fig.2).

Differentiating the second equality in (1.1) with respect to v, we obtain $d(u-u_0)^2 / dv = (v_0 - v)(dp / dv - (p-p_0) / (v_0 - v))$, whence $dp / dv > D^2$ in Ω_β, but since $dD^2 / ds > 0$, then in this region p increases strictly monotonically together with v, s and D^2.

Let us consider the wave adiabatic curve W when $v \geqslant v_0$. In this case a second constraint is imposed, namely, (1.5); therefore, one of the distinctive properties is the absence of turning points since at them $\partial p / \partial s = 2T / (v_0 - v)$, i.e., all admissible segments $H_s \subset \Omega_\beta$. The remaining properties can be established by analogy with the case $v \leqslant v_0$.

Theorem 4. The wave adiabatic curve W exists and is unique.

Proof. First of all we note that we are examining sufficiently smooth functions and can therefore exclude the possibility that an infinite number of Poisson adiabatic curves, occurring in W, exist in a bounded region. The existence of W follows at once from Theorems 1 and 3 and from the properties considered above. To prove the uniqueness it is necessary to show that H does not intersect P and the curves R when $v \in (v_3, v_1)$ (Fig.1). The discussions are carried out in the plane π_p.

1) Let us show that H_p does not intersect P_p. Assume the contrary: Let H_p and P_p intersect at a point B (Fig.3). Then we have

$$\varepsilon_B - \varepsilon_0 = SOBCE$$

$$\varepsilon_B - \varepsilon_A = \int_{v_C}^{v_D} p\,dv = S\widehat{ABC}D, \qquad \varepsilon_A - \varepsilon_0 = SOADE$$

$$SOBCE < S\widehat{ABC}D + SOADE$$

(here and further S denotes the area of the corresponding figure). Adding the last two equalities, we obtain a contradiction with the inequality.

(2) Let H_p intersect R_p at a point C (Fig.4). Then we have

$$\varepsilon_C - \varepsilon_0 = SOCDK$$

$$\varepsilon_C - \varepsilon_B = SBCDE, \qquad \varepsilon_B - \varepsilon_A = \int_{v_E}^{v_F} p\,dv = S\widehat{AB}EF$$

$$\varepsilon_A - \varepsilon_0 = SOAFK$$

$$SOCDK < SBCDE + S\widehat{AB}EF + SOAFK$$

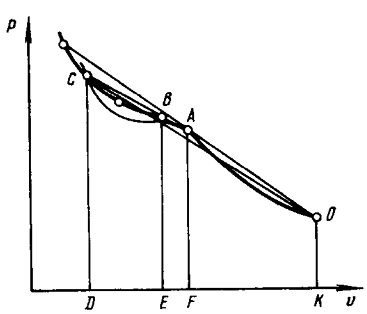

Fig.4

Adding the last three equalities, we obtain a contradiction with the inequality. The uniqueness of W follows from Theorems 1 and 3, the arguments presented, and the analogous arguments when $v \geqslant v_0$.

Wave adiabatic curves have the following properties.

Lemma 3. The wave adiabatic curve always has a break at the point (v_3, p_3) (Fig.1) when $ds \neq 0$.

Proof. Indeed

$$ds / dv = (-\partial p / \partial v - (p - \bar{p}) / (\bar{v} - v)) / (2T / (\bar{v} - v) - \partial p / \partial s), \; p \leqslant p_3$$

$$ds / dv = (-\partial p / \partial v - (p - p_0) / (v_0 - v)) / (2T / (v_0 - v) - \partial p / \partial s), \; p \geqslant p_3$$

At point (v_3, p_3) the numerators are equal, but the denominators are always unequal, i.e. $ds / dv\,|_{p=p_3-0} \neq ds / dv\,|_{p=p_3+0}$ when $ds \neq 0$.

Lemma 4. Turning points a (Fig.2) do not exist for media in which $\varepsilon = \Phi(v)\,\varphi(s)$, $\varepsilon_0 > 0$.

Proof. From (1.1) we have $p = 2\varepsilon / (v_0 - v) - 2\varepsilon_0 / (v_0 - v) - p_0$. Equalities (a), (e), (f) are fulfilled at point a. Having made the substitution $\partial p/\partial s = 2T / (v_0 - v)$, we obtain $pT - \varepsilon\partial p / \partial s = -2\varepsilon_0 T / (v_0 - v) - p_0 T$. Hence $pT - \varepsilon \partial p / \partial s < 0$. If $\varepsilon = \Phi(v)\,\varphi(s)$, then $pT = \varepsilon \partial p / \partial s$, which contradicts the last inequality.

Theorem 5. On the wave adiabatic W the entropy s increases monotonically with the increase of pressure when $v \leqslant v_0$ and with the decrease of pressure when $v \geqslant v_0$.

Proof. $dD^2/ds > 0$ holds everywhere on $H_p \subset W_p$ and $R_p \subset W_p$. Now by examining the disposition of H_p and R_p relative to the rays drawn from the point (v_0, p_0) and $(\bar{v}, \bar{p})$ in regions Ω_α, Ω_β, $\bar{\Omega}_\alpha$, $\bar{\Omega}_\beta$ when $v \leqslant v_0$, we can be convinced that $ds/dp \geqslant 0$, while, by definition, $ds/dp = 0$ on $P_p \subset W_p$. Analogously for $v \geqslant v_0$.

4. Uniqueness theorem. Theorem 6. The Riemann problem on the disintegration of an arbitrary discontinuity has a unique solution.

Proof. For the proof it is sufficient that the velocity u increase strictly monotonically with the growth of pressure p. Then in the plane (u, p) two wave adiabatic curves with initial states equal to the states on the initial discontinuity intersect at one point, and in this way a physical state is uniquely defined on the contact discontinuity /3/. The monotonic increase of the velocity when $v \leqslant v_0$ follows from Theorem 2 and the above-presented properties of wave adiabatic curves, while when $v \geqslant v_0$ it follows from the fact that the wave adiabatic curves do not have turning points, i.e., segments where the velocity's monotonicity could be violated.

5. Notes on inequalities (1.4), (1.5). As we see from the text, all the proofs of the fundamental properties of wave adiabatic curves are based on inequalities (1.4) and (1.5). For any medium defined by the general thermodynamic relations (1.2) we can always find a sufficiently small finite region containing (v_0, s_0, p_0), where (1.4) and (1.5) are obviously fulfilled; therefore, they are additional constraints only when considering wave adiabatic curves in the large. A wide class of media exists, for which it is easy to prove the fulfillment of (1.4) and (1.5) on wave adiabatic curves in the large. Indeed, in view of (1.1) inequality (1.4) is equivalent to the following: $\varepsilon < 2c_v T(p + p_0)/(p - p_0) + \varepsilon_0$. Hence

$$\int_0^T c_v|_{v=\text{const}}\, dT < 2c_v T(p + p_0)/(p - p_0) + \varepsilon_0 \tag{5.1}$$

where $v \geqslant v_k$, v_k is the medium's specific volume when $T = 0, p = 0$ ($v_k = 0$ for ideal media). From (5.1) we see the fulfillment of (1.4) for media in which the heat capacity is constant or grows with the growth of temperature. In addition, (5.1) also resolves the decrease of heat capacity c_v with growth of temperature, but sufficiently slowly. The complete domain in which (5.1) is fulfilled is obtained by investigating the dependence of c_v on the temperature.

Inequality (1.5) is always fulfilled for media in which $\partial p/\partial s > 0$. Let us discuss inequality (1.4) further. We rewrite the last condition in (1.3) as

$$-Tc_v^{-1}\partial p/\partial v = \alpha\,(\partial p/\partial s)^2 \tag{5.2}$$

where the coefficient $\alpha \geqslant 1$. In the region Ω_α, under the condition $-2T/(v_0 - v) < \partial p/\partial s$, from (5.2) we obtain an inequality equivalent to (1.4)

$$(p - p_0)(v_0 - v) < 4\alpha c_v T \tag{5.3}$$

We note that $\partial p/\partial s > 2T/(v_0 - v)$ in Ω_β and that it is not possible to derive (5.3) from (5.2) in this region. Other equivalent inequalities for (1.4) are $u^2 < 4c_v T$ when $u_0 = 0$ and $\oint T ds < 4c_v T$, where the integration is carried out over a rectangular contour with sides $v = v_0$, $p = p_0$ and the corresponding segments of the straight lines passing through (v, p).

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